Quantum Harmonic Oscillator — Complete Cheat Sheet

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❶ FUNDAMENTALS

▸ Canonical Quantization

$[\hat{x}, \hat{p}] = i\hbar$

Position rep: $\hat{p} = -i\hbar\,\partial/\partial x$

▸ The QHO Hamiltonian

$\hat{H} = \hat{p}^2/(2m) + m\omega^2\hat{x}^2/2$

$= \hbar\omega(\hat{N} + 1/2)$

$= \hbar\omega(\hat{a}^\dagger\hat{a} + 1/2)$

▸ Energy Eigenvalues

$E_n = \hbar\omega(n + 1/2), \quad n = 0, 1, 2, \ldots$

Zero-point: $E_0 = \hbar\omega/2$

Level spacing: $\Delta E = \hbar\omega$

▸ Natural Length Scale

$x_0 = \sqrt{\hbar/m\omega}$ (oscillator length)

$\xi = x/x_0$ (dimensionless)

Virial: $\langle T\rangle_n = \langle V\rangle_n = E_n/2$

❷ LADDER OPERATORS

▸ Definitions

$\hat{a} = \sqrt{m\omega/2\hbar}\,(\hat{x} + i\hat{p}/m\omega)$

$\hat{a}^\dagger = \sqrt{m\omega/2\hbar}\,(\hat{x} - i\hat{p}/m\omega)$

▸ Inverse Relations

$\hat{x} = (x_0/\sqrt{2})(\hat{a} + \hat{a}^\dagger)$

$\hat{p} = i\hbar/(\sqrt{2}\,x_0)(\hat{a}^\dagger - \hat{a})$

Number op: $\hat{N} = \hat{a}^\dagger\hat{a}$, $\hat{N}|n\rangle = n|n\rangle$

▸ Commutators

$[\hat{a}, \hat{a}^\dagger] = 1$

$[\hat{N}, \hat{a}] = -\hat{a}, \quad [\hat{N}, \hat{a}^\dagger] = +\hat{a}^\dagger$

$[\hat{H}, \hat{a}] = -\hbar\omega\hat{a}$

$[\hat{H}, \hat{a}^\dagger] = +\hbar\omega\hat{a}^\dagger$

▸ Action on Fock States

$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$

$\hat{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$

$|n\rangle = (\hat{a}^\dagger)^n/\sqrt{n!}\,|0\rangle$

$\hat{a}|0\rangle = 0$ (ground state)

▸ Matrix Elements

$\langle m|\hat{x}|n\rangle = (x_0/\sqrt{2})[\sqrt{n}\,\delta_{m,n-1} + \sqrt{n+1}\,\delta_{m,n+1}]$

$\langle n|\hat{x}^2|n\rangle = x_0^2(n + 1/2)$

$\langle n|\hat{p}^2|n\rangle = (\hbar^2/x_0^2)(n + 1/2)$

❸ WAVEFUNCTIONS

▸ General Formula

$\psi_n(x) = (m\omega/\pi\hbar)^{1/4}$

$\times H_n(\xi)/\sqrt{2^n n!}\cdot e^{-\xi^2/2}$

▸ Ground State (Gaussian)

$\psi_0(x) = (\pi x_0^2)^{-1/4}\exp(-x^2/2x_0^2)$

(Minimum uncertainty — no nodes)

▸ Hermite Polynomials $H_n(\xi)$

$H_0 = 1$

$H_1 = 2\xi$

$H_2 = 4\xi^2 - 2$

$H_3 = 8\xi^3 - 12\xi$

$H_4 = 16\xi^4 - 48\xi^2 + 12$

Recurrence: $H_{n+1} = 2\xi H_n - 2nH_{n-1}$

Derivative: $H_n' = 2nH_{n-1}$

▸ Key Properties of ψ_n

Parity: $\psi_n(-x) = (-1)^n\psi_n(x)$

Nodes: ψ_n has exactly n zeros

Ortho: $\int\psi_m^*\psi_n\,dx = \delta_{mn}$

▸ Expectation Values in $|n\rangle$

$\langle\hat{x}\rangle = 0, \quad \langle\hat{p}\rangle = 0$

$\langle\hat{x}^2\rangle = x_0^2(n + 1/2)$

$\Delta x = x_0\sqrt{n + 1/2}$

$\Delta x\,\Delta p = (n + 1/2)\hbar \geq \hbar/2$

Turning pts: $x_\pm = \pm\sqrt{2n+1}\,x_0$

Tunneling prob. for n=0: ~15.7%

❹ TIME EVOLUTION

▸ Schr\"{o}dinger Picture

$|n, t\rangle = e^{-i\omega(n+1/2)t}|n\rangle$

$|\Psi(t)\rangle = \sum_n c_n e^{-i\omega(n+1/2)t}|n\rangle$

Period of full revival: $T = 2\pi/\omega$

▸ Heisenberg Picture

$\hat{a}(t) = e^{-i\omega t}\hat{a}(0)$

$\hat{a}^\dagger(t) = e^{+i\omega t}\hat{a}^\dagger(0)$

$\hat{x}(t) = x_0\cos\omega t\,\hat{x}(0)$

$+\sin(\omega t)/(m\omega)\,\hat{p}(0)$

▸ Ehrenfest's Theorem

$d\langle\hat{x}\rangle/dt = \langle\hat{p}\rangle/m$

$d\langle\hat{p}\rangle/dt = -m\omega^2\langle\hat{x}\rangle$

$\Rightarrow\ d^2\langle\hat{x}\rangle/dt^2 = -\omega^2\langle\hat{x}\rangle$

Valid for ANY quantum state!

▸ Wigner Function

$W(x,p) = \frac{1}{\pi\hbar}\int\psi^*(x+y)\psi(x-y)e^{2ipy/\hbar}dy$

$\int W\,dp = |\psi(x)|^2, \ \int W\,dx = |\phi(p)|^2$

$W_0 = \frac{1}{\pi\hbar}e^{-x^2/x_0^2 - p^2x_0^2/\hbar^2}$

W < 0 possible → non-classical signature

❺ COHERENT STATES

▸ Definition

$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle, \quad \alpha\in\mathbb{C}$

(eigenstate of annihilation operator)

▸ Fock-State Expansion

$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$

▸ Displacement Operator

$\hat{D}(\alpha) = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}$

$= e^{-|\alpha|^2/2}e^{\alpha\hat{a}^\dagger}e^{-\alpha^*\hat{a}}$ (BCH)

$|\alpha\rangle = \hat{D}(\alpha)|0\rangle$

$\hat{D}^\dagger\hat{a}\hat{D} = \hat{a} + \alpha$

▸ BCH Lemma

If $[A,[A,B]] = [B,[A,B]] = 0$:

$e^{A+B} = e^A e^B e^{-[A,B]/2}$

▸ Expectation Values

$\langle\hat{x}\rangle = \sqrt{2}\,x_0\,\mathrm{Re}(\alpha)$

$\langle\hat{p}\rangle = (\sqrt{2}\,\hbar/x_0)\,\mathrm{Im}(\alpha)$

$\Delta x = x_0/\sqrt{2}$ (independent of α!)

$\Delta p = \hbar/(\sqrt{2}\,x_0)$ (independent of α!)

$\Delta x\,\Delta p = \hbar/2$ \u2190 MINIMUM

▸ Photon Number

$P(n) = e^{-|\alpha|^2}|\alpha|^{2n}/n!$ [Poisson]

$\langle\hat{N}\rangle = |\alpha|^2, \quad \Delta N = |\alpha|$

Mandel Q = 0 (classical boundary)

▸ Time Evolution

$\hat{U}(t)|\alpha\rangle = e^{-i\omega t/2}|\alpha e^{-i\omega t}\rangle$

Stays coherent! $\alpha\to\alpha e^{-i\omega t}$

$\langle\hat{x}\rangle(t) = \sqrt{2}\,x_0|\alpha|\cos(\omega t - \phi)$

▸ Overlaps & Over-Completeness

$|\langle\beta|\alpha\rangle|^2 = e^{-|\alpha-\beta|^2}$

$(1/\pi)\int|\alpha\rangle\langle\alpha|\,d^2\alpha = \hat{I}$ (over-complete)

❽ QUICK COMPARISON

State	Δx	Δp	$\Delta x\Delta p$
Fock $\lvert n\rangle$	$x_0\sqrt{n+\frac{1}{2}}$	$\frac{\hbar}{x_0}\sqrt{n+\frac{1}{2}}$	$\hbar(n+\frac{1}{2})$
Coherent $\lvert\alpha\rangle$	$x_0/\sqrt{2}$	$\hbar/\sqrt{2}\,x_0$	$\hbar/2$
Squeezed	$x_0e^{-r}/\sqrt{2}$	$\hbar e^r/\sqrt{2}\,x_0$	$\hbar/2$

❻ SQUEEZED STATES

▸ Quadrature Operators

$\hat{X}_1 = (\hat{a} + \hat{a}^\dagger)/2, \quad \hat{X}_2 = (\hat{a} - \hat{a}^\dagger)/2i$

$[\hat{X}_1, \hat{X}_2] = i/2, \quad \Delta X_1\Delta X_2 \geq 1/4$

Coherent: $\Delta X_1 = \Delta X_2 = 1/2$

▸ Squeeze Operator

$\hat{S}(\xi) = \exp(\xi^*\hat{a}^2/2 - \xi\hat{a}^{\dagger 2}/2),\ \xi = re^{i\theta}$

$\hat{S}^\dagger = \hat{S}^{-1} = \hat{S}(-\xi)$ [unitary]

▸ Bogoliubov Transformation

$\hat{S}^\dagger(r)\hat{a}\hat{S}(r) = \hat{a}\cosh r - \hat{a}^\dagger\sinh r$

$\hat{S}^\dagger(r)\hat{a}^\dagger\hat{S}(r) = \hat{a}^\dagger\cosh r - \hat{a}\sinh r$

New ops: $[\hat{b}, \hat{b}^\dagger] = 1$ \u2713

▸ Squeezed Vacuum $|0, r\rangle = \hat{S}(r)|0\rangle$

$\Delta x = x_0\,e^{-r}/\sqrt{2}$ \u2190 below SQL!

$\Delta p = \hbar\,e^{+r}/(\sqrt{2}\,x_0)$ \u2190 above SQL

$\Delta x\,\Delta p = \hbar/2$ \u2190 still minimum!

$\langle\hat{N}\rangle = \sinh^2 r$ (photons in vacuum!)

▸ Fock expansion (even n only)

$|0, r\rangle = \frac{1}{\sqrt{\cosh r}}\sum_n\frac{(-\tanh r)^n\sqrt{(2n)!}}{2^n n!}|2n\rangle$

▸ Squeezed Coherent State

$|\alpha, \xi\rangle = \hat{D}(\alpha)\hat{S}(\xi)|0\rangle$

Mandel Q < 0 → sub-Poissonian (non-classical)

❼ LIGO & APPLICATIONS

LIGO measures phase quadrature $\hat{X}_2$

Shot noise (SQL): $\Delta X_2 = 1/2$

Phase-squeezed input: $\Delta X_2 \to \Delta X_2\,e^{-r}$

SNR improvement factor: e^r

LIGO O3 (2019): $r \approx 0.5$\u20131.0 (4\u20139 dB)

→ 50% improvement in detection range

Applications: CV-QKD, quantum teleportation,

Gaussian boson sampling, spin squeezing

❾ KEY IDENTITIES

▸ Gaussian Integrals

$\int_{-\infty}^{\infty}e^{-\alpha x^2}dx = \sqrt{\pi/\alpha}$

$\int_{-\infty}^{\infty}x^{2n}e^{-x^2}dx = (2n-1)!!/(2^n)\sqrt{\pi}$

▸ Hermite Orthogonality

$\int H_m H_n e^{-\xi^2}d\xi = 2^n n!\sqrt{\pi}\,\delta_{mn}$

▸ Uncertainty Relations

Heisenberg: $\Delta x\,\Delta p \geq \hbar/2$

Number-phase: $\Delta N\,\Delta\phi \geq 1/2$

Time-energy: $\Delta E\,\Delta t \geq \hbar/2$

▸ Commutator Rules

$[\hat{A}, \hat{B}\hat{C}] = [\hat{A}, \hat{B}]\hat{C} + \hat{B}[\hat{A}, \hat{C}]$

$[\hat{a}, f(\hat{a}^\dagger)] = f'(\hat{a}^\dagger)$ for analytic f

Shankar — Ch. 7 | Griffiths — §2.3

Cohen-Tannoudji Vol. 1 Ch. V + Compl. GV

Walls & Milburn — Quantum Optics

PhysicsVoyage → Courses → QHO